A Zero-Pole-Gain State-Space Realization for Single-Input-Single-Output Transfer Functions

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***Abstract*— This paper demonstrates the Pole-Zero-Difference Form of state-space realization for all proper transfer functions, where all the distinct, real poles, zeros and gain of the transfer function appear as explicit components in the state space.**

I. Introduction

Transfer functions can be expressed in multiple different state-space realizations in order to optimize their utility in different applications. There are several so-called canonical realizations that appear in the literature. The two most common, the *Observable* and *Controllable Forms*, are expressed utilizing the coefficients of the transfer functions, a_i and b_i, when expressed as polynomials, such as;

$$\frac{Y(s)}{U(s)} = \frac{b_1 s^{n-1} + \ldots + b_{n-1}s + b_n}{s^n + a_1 s^{n-1} + \ldots + a_{n-1}s + a_n} \tag{1}$$

Furthermore, the *Diagonal Form* can be used when the denominator can be factorized into distinct poles;

$$\frac{Y(s)}{U(s)} = \frac{b_1 s^{n-1} + \ldots + b_{n-1}s + b_n}{(s-p_1)(s-p_2)...(s-p_n)} \tag{2}$$

Where $p_1 \neq p_2 \neq ... \neq p_n$. In order to then construct the state-space for this form, the partial fractions need to be found, such that;

$$\frac{Y(s)}{U(s)} = \frac{r_1}{(s-p_1)} + \frac{r_2}{(s-p_2)} + ... + \frac{r_n}{(s-p_n)} \tag{3}$$

Thus the *Diagonal Form* state-space can be expressed using the coefficients r_i and poles p_i. However, the literature has no such method to readily express a state-space realization explicitly using the poles p_i, zeros z_i and gain k of the transfer function, i.e. where both the numerator and denominator have been factorized;

$$\frac{Y(s)}{U(s)} = \frac{(s-z_1)(s-z_2)...(s-z_{n-1})}{(s-p_1)(s-p_2)...(s-p_n)}k \tag{4}$$

Indeed, the above is true for any number of zeros up to n (where the original equation would have the additional term $b_0 s^n$ in the numerator). This paper presents a new state-space realization, the *Pole-Zero-Difference Form*, such that this is possible. Below follows (II) a problem statement, (III) the solution for all cases , (IV) proof of the solution, (IV-A) reference of simple cases, (IV-B) and (IV-C) comparisons with other canonical forms, and (IV-D) Matlab example.

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II. Problem Statement

Consider the single-input-single-output transfer function in pole-zero form;

$$\frac{Y(s)}{U(s)} = \frac{\prod_{i=1}^{M}(s-z_i)}{\prod_{j=1}^{N}(s-p_j)}.k \tag{5}$$

Where $1 \leq M \leq N$, and the values of p_j and z_i are distinct and unique. We seek to construct a state-space realization of this function such that the poles p_j, zeros z_i and gain k appear explicitly in the matrices of $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$ and $\mathbf{D}$, thus;

$$\begin{aligned} \dot{\boldsymbol{x}}(t) &= \mathbf{A}\boldsymbol{x}(t) + \mathbf{B}u(t) \\ y(t) &= \mathbf{C}\boldsymbol{x}(t) + \mathbf{D}u(t) \end{aligned} \tag{6}$$

The arbitrary simple case of $M = 0$ and numerator equals 1 is also considered.

III. Overview of Solution

A. The 4 Main Categories

The *Pole-Zero-Difference Form* is named for the off-diagonal terms in the $\mathbf{A}$ matrix. Realizations in this form are presented for all proper transfer functions. However, the algebra becomes incredibly expansive to prove in the general case. As such, this paper will explore the solution across a number of cases in order to build the methodology. There is significant overlap between these categories, with most of the differences confined to the $\mathbf{A}$ matrix. There are 4 categories - functions where;

1) there with no zeros, $M = 0$.
2) the number of zeros is anything up to two less than the number of poles, $0 < M \leq N - 2$.
3) the number of zeros is exactly one less than that of poles, $M = N - 1$.
4) the number of zeros and poles is equal, $M = N$.

Matrix $\mathbf{A}$ will always be square and of size $N \times N$, and it follows that $\mathbf{B}$ will be a column of length N, $\mathbf{C}$ is a row of length N, and $\mathbf{D}$ will have a single element. In all categories, Matrix $\mathbf{B}$ will be constructed;

$$\mathbf{B} = \begin{bmatrix} k & 0 & 0 & \ldots & 0 \end{bmatrix}^{\mathsf{T}} \tag{7}$$

and in the cases where $N = 1$,

$$\mathbf{B} = \begin{bmatrix} k \end{bmatrix} \tag{8}$$

Similarly, matrix $\mathbf{C}$ is a row of length N. For categories 1 and 2;

$$\mathbf{C} = \begin{bmatrix} 0 & \ldots & 0 & 0 & 1 \end{bmatrix} \tag{9}$$

and for category 3;

$$\mathbf{C} = \begin{bmatrix} 1 & \dots & 1 & 1 & 1 \end{bmatrix} \tag{10}$$

For category 4, $\mathbf{C}$ is more complicated, and the details will be explored further below;

$$\mathbf{C} = \begin{bmatrix} \sum_{i=1}^{N}(p_i - z_i) & \dots & \sum_{i=N-1}^{N}(p_i - z_i) & p_N - z_N \end{bmatrix} \tag{11}$$

Finally, $\mathbf{D} = 0$ for categories 1-3 and $\mathbf{D} = k$ for category 4. The example cases below will revisit each of these categories.

B. The construction of $\mathbf{A}$

The matrix $\mathbf{A}$ is always lower triangular, and constructed by placing the poles (p_1 to p_N) on the main diagonal of a square matrix. For the case of no zeros, 1's are placed on the subdiagonal, one place to the left of the main. For example;

$$\mathbf{A} = \begin{bmatrix} p_1 & 0 & 0 & 0 & \dots & 0 \\ 1 & p_2 & 0 & 0 & & \\ 0 & 1 & p_3 & 0 & & \\ 0 & 0 & 1 & p_4 & & \\ \vdots & & & \ddots & \ddots & 0 \\ 0 & & \dots & 0 & 1 & p_N \end{bmatrix} \tag{12}$$

Where the full state-space would be;

$$\begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{bmatrix} = \begin{bmatrix} p_1 & 0 & 0 & 0 & \dots & 0 & k \\ 1 & p_2 & 0 & 0 & & & 0 \\ 0 & 1 & p_3 & 0 & & & 0 \\ 0 & 0 & 1 & p_4 & & & 0 \\ \vdots & & & \ddots & \ddots & 0 & 0 \\ 0 & & \dots & 0 & 1 & p_N & 0 \\ 0 & & \dots & 0 & 0 & 1 & 0 \end{bmatrix} \tag{13}$$

This would represent the transfer function;

$$Y(s) = \frac{1}{\prod_{j=1}^{N}(s - p_i)}.kU(s) \tag{14}$$

This particular form is already well known, and represents the complete set of functions from category 1 from the overview. It is trivial to prove so not considered further.

We next consider simply $M = 1$, with the zero z_1. The zero is placed in $\mathbf{A}$ on the second row under the first pole, as the pole-zero difference (p_2-z_1). Additionally, the third row (the $M + 2^{th}$ row) will now have 1's up until the main diagonal;

$$\mathbf{A} = \begin{bmatrix} p_1 & 0 & 0 & 0 & \dots & 0 \\ p_2 - z_1 & p_2 & 0 & 0 & & \\ 1 & 1 & p_3 & 0 & & \\ 0 & 0 & 1 & p_4 & & \\ \vdots & & & \ddots & \ddots & 0 \\ 0 & & \dots & 0 & 1 & p_N \end{bmatrix} \tag{15}$$

For a second zero, z_2, the pattern continues, but now there are two columns with the pole-zero difference (p_3-z_2). Again, the $M + 2^{th}$ row will have 1's up until the main diagonal;

$$\mathbf{A} = \begin{bmatrix} p_1 & 0 & 0 & 0 & \dots & 0 \\ p_2 - z_1 & p_2 & 0 & 0 & & \\ p_3 - z_2 & p_3 - z_2 & p_3 & 0 & & \\ 1 & 1 & 1 & p_4 & & \\ \vdots & & & \ddots & \ddots & 0 \\ 0 & & \dots & 0 & 1 & p_N \end{bmatrix} \tag{16}$$

So, for each subsequent zero up to z_M, the rows are constructed as;

$$\mathbf{A} = \begin{bmatrix} p_1 & & & & & & & \\ p_2 - z_1 & p_2 & & & & & & \\ \vdots & & \ddots & & & & & \\ & & & [- & \mathbf{A}_{M+1} & -] & & \\ 1 & \dots & 1 & 1 & p_{M+2} & & & \\ 0 & & \dots & 0 & 1 & p_{M+3} & & \\ \vdots & & & & & & \ddots & \\ 0 & & \dots & & & 0 & 1 & p_N \end{bmatrix} \tag{17}$$

where the $\mathbf{A}_{M+1}$ row;

$$\mathbf{A}_M = \begin{bmatrix} p_{M+1} - z_M & \dots & p_{M+1} - z_M & p_{M+1} & 0 & \dots & 0 \end{bmatrix}$$

This pattern continues for all category 2 cases (up to $M = N - 2$), where the final realization is mostly the same but with no 0's left beneath the diagonal;

$$\mathbf{A} = \begin{bmatrix} p_1 & & & & \\ p_2 - z_1 & p_2 & & & \\ p_3 - z_2 & p_3 - z_2 & p_3 & & \\ \vdots & & & \ddots & \\ p_{N-1} - z_M & \dots & p_{N-1} - z_M & p_{N-1} & \\ 1 & \dots & 1 & 1 & p_N \end{bmatrix} \tag{18}$$

However, for category 3 where $M = N - 1$, there is no room for the row of 1's, so;

$$\mathbf{A} = \begin{bmatrix} p_1 & & & \\ p_2 - z_1 & p_2 & & \\ p_3 - z_2 & p_3 - z_2 & p_3 & \\ \vdots & & & \ddots \\ p_N - z_M & \dots & p_N - z_M & p_N \end{bmatrix} \tag{19}$$

And this is why the row of 1's is shifted to $\mathbf{C}$, as discussed above. Now, $\mathbf{C}$ is effectively $M + 2^{th}$ row which is an important row to consider when constructing the proof.

C. The construction of $\mathbf{A}$*,* $\mathbf{C}$ *and* $\mathbf{D}$ *for category 4*

For the case where $M = N$, matrix $\mathbf{A}$ is unchanged from that shown for category 3 (See equation 19) and matrix $\mathbf{D}$ must be nonzero and will always be $\mathbf{D} = k$. For matrix $\mathbf{C}$, each element of the row will now be a summation of pole-zero differences. Be aware that these are slightly different to the pole-zero differences in $\mathbf{A}$. By way of example, consider the case of $N = M = 1$;

$$\begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{bmatrix} = \begin{bmatrix} p_1 & k \\ p_1 - z_1 & k \end{bmatrix} \tag{20}$$

For each ith additional pole and zero, $\mathbf{C}$ expands by 1 column, and $p_i - z_i$ is added to each element. For instance, for $N = M = 2$;

$$\begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{bmatrix} = \begin{bmatrix} p_1 & 0 & k \\ p_2 - z_1 & p_2 & 0 \\ p_1 - z_1 + p_2 - z_2 & p_2 - z_2 & k \end{bmatrix} \tag{21}$$

or in general, for a function of N poles and zeros;

$$\mathbf{C} = \begin{bmatrix} \sum_{i=1}^{N}(p_i - z_i) & \dots & \sum_{i=N-1}^{N}(p_i - z_i) & p_N - z_N \end{bmatrix} \tag{22}$$

For visual clarity as used in the Appendices, we can introduce the term Δ_{ji} where $\Delta = p_j - z_i$. With this, the equations above 20, 21 and 22 could be rewritten, respectively, as;

$$\begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{bmatrix} = \begin{bmatrix} p_1 & k \\ \Delta_{11} & k \end{bmatrix} \tag{23}$$

$$\begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{bmatrix} = \begin{bmatrix} p_1 & 0 & k \\ \Delta_{21} & p_2 & 0 \\ \Delta_{11} + \Delta_{22} & \Delta_{22} & k \end{bmatrix} \tag{24}$$

$$\mathbf{C} = \begin{bmatrix} \sum_{i=1}^{N} \Delta_{ii} & \dots & \sum_{i=N-1}^{N} \Delta_{ii} & \Delta_{NN} \end{bmatrix} \tag{25}$$

We need to inspect the individual p_j and z_i in order to prove these solutions, so the use of Δ_{ji} will not be included below, but it may be useful for the reader outside of this paper.

IV. Solving the Laplacian

The problem with finding the general solution to these realizations is that the edge cases don't always fit neatly into the regular algebraic patterns. It is easier to work through a number of examples in order to familiarize oneself with the manipulations. These following examples will cover cases from Category 2, then 3 and then 4.

Consider solving the family of differential equations that come from the state-space constructed using Equation 17, such that $M >> 0$, and $N >> M$.

$$\begin{aligned} \dot{\boldsymbol{x}}_1 &= p_1\boldsymbol{x}_1 + ku \\ \dot{\boldsymbol{x}}_2 &= (p_2 - z_1)\boldsymbol{x}_1 + p_2\boldsymbol{x}_2 \\ \dot{\boldsymbol{x}}_3 &= (p_3 - z_2)[\boldsymbol{x}_1 + \boldsymbol{x}_2] + p_3\boldsymbol{x}_3 \\ &\vdots \\ \dot{\boldsymbol{x}}_{M+1} &= (p_{M+1} - z_M)[\boldsymbol{x}_1 + \ldots + \boldsymbol{x}_M] + p_{M+1}\boldsymbol{x}_{M+1} \\ \dot{\boldsymbol{x}}_{M+2} &= [\boldsymbol{x}_1 + \ldots + \boldsymbol{x}_M] + p_{M+2}\boldsymbol{x}_{M+2} \\ \dot{\boldsymbol{x}}_{M+3} &= \boldsymbol{x}_{M+2} + p_{M+3}\boldsymbol{x}_{M+3} \\ &\vdots \\ \dot{\boldsymbol{x}}_N &= \boldsymbol{x}_{N-1} + p_N\boldsymbol{x}_N \\ y &= \boldsymbol{x}_N \end{aligned} \tag{26}$$

If we take the Laplace transform of each equation, we get;

$$\begin{aligned} X_1 s &= p_1 X_1 + kU \\ X_2 s &= (p_2 - z_1)X_1 + p_2 X_2 \\ X_3 s &= (p_3 - z_2)[X_1 + X_2] + p_3 X_3 \\ &\vdots \\ X_{M+1} s &= (p_{M+1} - z_M)[X_1 + \ldots + X_M] + p_{M+1} X_{M+1} \\ X_{M+2} s &= [X_1 + \ldots + X_{M+1}] + p_{M+2} X_{M+2} \\ X_{M+3} s &= X_{M+2} + p_{M+3} X_{M+3} \\ &\vdots \\ X_N s &= X_{N-1} + p_N X_N \\ Y &= X_N \end{aligned} \tag{27}$$

Each equation can then be rearranged to give a specific row by row solution;

$$\begin{aligned} X_1 &= \frac{1}{(s - p_1)} kU \\ X_2 &= \frac{(p_2 - z_1)[X_1]}{(s - p_2)} \\ X_3 &= \frac{(p_3 - z_2)[X_1 + X_2]}{(s - p_3)} \\ &\vdots \\ X_{M+1} &= \frac{(p_{M+1} - z_M)[X_1 + \ldots + X_M]}{(s - p_{M+1})} \\ X_{M+2} &= \frac{[X_1 + \ldots + X_{M+1}]}{(s - p_{M+2})} \\ X_{M+3} &= \frac{X_{M+2}}{(s - p_{M+3})} \\ &\vdots \\ X_N &= \frac{X_{N-1}}{(s - p_N)} \\ Y &= X_N \end{aligned} \tag{28}$$

We contend that this will eventually lead back to the regular pole-zero form shown in Equation 5. However, to prove this, we should consider a few simple cases before exploring the full argument, because the patterns are not straightforward.

Case 1, 2 poles, 1 zero

Firstly, let $N = 2$ and $M = 1$. Because $M = N - 1$, $\mathbf{C} = [11]$, In this case, the family of $N + 1 = 3$ equations will be simply;

$$\begin{aligned} X_1 &= \frac{1}{(s-p_1)}kU \\ X_2 &= \frac{(p_2-z_1)[X_1]}{(s-p_2)} \\ Y &= X_2 + X_1 \end{aligned} \tag{29}$$

Firstly, substituting X_1 into X_2;

$$X_2 = \frac{(p_2-z_1)}{(s-p_2)(s-p_1)}kU \tag{30}$$

Then Y can be solved easily by substitution;

$$Y = \frac{(p_2-z_1)}{(s-p_2)(s-p_1)}kU + \frac{1}{(s-p_1)}kU \tag{31}$$

Cross multiply for a common denominator;

$$Y = \frac{(p_2-z_1)}{(s-p_2)(s-p_1)}kU + \frac{(s-p_2)}{(s-p_2)(s-p_1)}kU \tag{32}$$

Simplify the numerator;

$$Y = \frac{(p_2-z_1)+(s-p_2)}{(s-p_2)(s-p_1)}kU \tag{33}$$

$$Y = \frac{(s-z_1)}{(s-p_2)(s-p_1)}kU \tag{34}$$

This solution matches the form given in equation 5, and thus works as a proof of that case. What we would like to show is that the poles in the numerator will disappear no matter how many equations. We can show that this cancellation will always occur in the equation for X_{M+2} (except in the case where $M = N - 1$, which is shown in Section IV, and the case $M = N$, shown in Section IV).

Case 2, 5 poles, 1 zero

Consider a second simple case, where $N = 5$, $M = 1$, thus $\mathbf{C} = [00001]$;

$$\begin{aligned} X_1 &= \frac{1}{(s-p_1)}kU \\ X_2 &= \frac{(p_2-z_1)[X_1]}{(s-p_2)} \\ X_3 &= \frac{[X_1+X_2]}{(s-p_3)} \\ X_4 &= \frac{X_3}{(s-p_4)} \\ X_5 &= \frac{X_4}{(s-p_5)} \\ Y &= X_5 \end{aligned} \tag{35}$$

There are still $N + 1$ equations. However, this time note that the last equation to introduce a new zero term is equation $M + 1$, and the summation of previous substitutions is in equation for X_{M+2} (i.e. the equation for X_3). The square brackets of this term will be the same as the final solution in Equation 34 (Section IV), and therefore;

$$\begin{aligned} X_3 &= \frac{1}{(s-p_3)}\frac{(s-z_1)}{(s-p_2)(s-p_1)}kU \\ X_4 &= \frac{X_3}{(s-p_4)} \\ X_5 &= \frac{X_3}{(s-p_5)} \\ Y &= X_5 \end{aligned} \tag{36}$$

It is trivial to do the substitutions for each line and show that the final solution is

$$Y = \frac{(s-z_1)}{(s-p_5)(s-p_4)(s-p_3)(s-p_2)(s-p_1)}kU \tag{37}$$

Which can be written as;

$$Y = \frac{(s-z_1)}{\prod_{i=1}^{5}(s-p_i)}kU \tag{38}$$

Once again the solution is in the target form of equation 5. It can also be observed that increasing the number of poles, and thus the number of equations, through N will not make the final substitution any more complicated. Thus, we need to ensure that the first $M + 2$ equations can produce the correct numerator for the final solution.

Case 3, 5 poles, 3 zeros

Consider the third example case in order to note the pattern of the development of the numerator of X_{M+2}. Consider $N = 5$, $M = 3$, thus $\mathbf{C} = [00001]$;

$$\begin{aligned} X_1 &= \frac{1}{(s-p_1)}kU \\ X_2 &= \frac{(p_2-z_1)}{(s-p_2)}[X_1] \\ X_3 &= \frac{(p_3-z_2)}{(s-p_3)}[X_1+X_2] \\ X_4 &= \frac{(p_4-z_3)}{(s-p_4)}[X_1+X_2+X_3] \\ X_5 &= \frac{1}{(s-p_5)}[X_1+X_2+X_3+X_4] \\ Y &= X_5 \end{aligned} \tag{39}$$

Taking what we know from previous examples, X_2 simplifies to;

$$X_2 = \frac{(p_2-z_1)}{(s-p_2)(s-p_1)}kU \tag{40}$$

We know $[X_1 + X_2]$ from Equation 34, thus X_3;

$$X_3 = \frac{(p_3-z_2)}{(s-p_3)}[\frac{(s-z_1)}{(s-p_2)(s-p_1)}kU] \tag{41}$$

$$X_3 = \frac{(p_3-z_2)(s-z_1)}{(s-p_2)(s-p_1)(s-p_3)}kU \tag{42}$$

Consider X_4;

$$X_4 = \frac{(p_4 - z_3)}{(s-p_4)} \times \Big[\frac{1}{(s-p_1)}kU + \frac{(p_2 - z_1)}{(s-p_2)(s-p_1)}kU + \frac{(p_3-z_2)(s-z_1)}{(s-p_2)(s-p_1)(s-p_3)}kU\Big] \quad (43)$$

Create the common denominator of the square brackets;

$$X_4 = \frac{(p_4 - z_3)}{(s-p_4)} \times \Big[\frac{(s-p_3)(s-p_2)}{(s-p_3)(s-p_2)(s-p_1)}kU + \frac{(s-p_3)(p_2-z_1)}{(s-p_3)(s-p_2)(s-p_1)}kU + \frac{(p_3-z_2)(s-z_1)}{(s-p_2)(s-p_1)(s-p_3)}kU\Big] \quad (44)$$

Factorise;

$$X_4 = \frac{(p_4-z_3)}{\prod_{j=1}^{4}(s-p_j)}kU \times \Big[(s-p_3)(s-p_2) + (s-p_3)(p_2-z_1) + (p_3-z_2)(s-z_1)\Big] \quad (45)$$

Simplifying the square brackets is trivial, but we are interested in establishing a pattern to solve the general case. Consider that there are three terms within the square brackets. Notice that only the first two contain combinations of p_2. We can eliminate that first. Start by expanding only the brackets that contain p_2;

$$\Big[(s-p_3)s - (s-p_3)p_2 + (s-p_3)p_2 - (s-p_3)z_1 + (p_3-z_2)(s-z_1)\Big] \quad (46)$$

All of the p_2 terms cancel, and the remaining $(s - p_3)$ terms can factorise;

$$\Big[(s-p_3)(s-z_1) + (p_3-z_2)(s-z_1)\Big] \quad (47)$$

Repeating the process by expanded the brackets containing p_3, cancelling, and refactorising, the square brackets reduce to;

$$\Big[(s-z_2)(s-z_1)\Big] \quad (48)$$

And therefore;

$$X_4 = \frac{(p_4-z_3)(s-z_2)(s-z_1)}{\prod_{j=1}^{4}(s-p_j)}kU \quad (49)$$

In this case, we are most interested in the equation for X_5, which is the X_{M+2} equation. If we write in out in full, finding the common denominator of the square brackets as we did above, we get;

$$X_5 = \frac{1}{\prod_{j=1}^{5}(s-p_j)}kU \times ... \Big[(s-p_3)(s-p_2)(s-p_1) + (p_2-z_1)(s-p_4)(s-p_3) + (p_3-z_2)(s-p_4)(s-z_1) + (p_4-z_3)(s-z_2)(s-z_1)\Big] \quad (50)$$

From this step, we can simplify the square brackets by expanding, cancelling, and refactorising as we did in Equations 46 and 47. This would give us the final solution in the form;

$$Y = \frac{\prod_{i=1}^{3}(s-z_i)}{\prod_{j=1}^{5}(s-p_i)}.kU \quad (51)$$

More importantly,we can observe that finally a pattern is emerging for finding a general equation for X_{M+2}.

Case 4, N poles, M zeroes, where $M < N - 1$

Using equation 50 as a base, we can generalise the equation for X_{M+2} to;

$$X_{M+2} = \frac{kU}{\prod_{j=1}^{M+2}(s-p_j)} \times \Big[\prod_{i=2}^{M+1}(s-p_i) + (p_2-z_1)\prod_{i=3}^{M+1}(s-p_i) + (p_3-z_2)\prod_{i=4}^{M+1}(s-p_i).(s-z_1) + (p_4-z_3)\prod_{i=5}^{M+1}(s-p_i).\prod_{i=1}^{2}(s-z_i) + \ldots + (p_M - z_{M-1})(s-p_{M+1}).\prod_{i=1}^{M-2}(s-z_i) + (p_{M+1}-z_M)\prod_{i=1}^{M-1}(s-z_i)\Big] \quad (52)$$

We can simplify the square brackets using the techniques used in Equations 46 and 47 above. As an example, consider just the first two terms;

$$\Big[\prod_{i=2}^{M+1}(s-p_i) + (p_2-z_1)\prod_{i=3}^{M+1}(s-p_i) + \ldots\Big] \quad (53)$$

And rearrange the first term to show $(s - p_2)$ explicitly;

$$\Big[(s-p_2)\prod_{i=3}^{M+1}(s-p_i) + (p_2-z_1)\prod_{i=3}^{M+1}(s-p_i) + \ldots\Big] \quad (54)$$

Expand the brackets containing p_2;

$$\Big[(s)\prod_{i=3}^{M+1}(s-p_i) - (p_2)\prod_{i=3}^{M+1}(s-p_i) + (p_2)\prod_{i=3}^{M+1}(s-p_i) - (z_1)\prod_{i=3}^{M+1}(s-p_i) + \ldots\Big] \quad (55)$$

All the terms with p_2 cancel, and the rest of the terms can be factorised;

$$\Big[\prod_{i=3}^{M+1}(s-p_i).(s-z_1) + \ldots\Big] \quad (56)$$

If we factor out the term $(s-p_3)$, and consider one more term from within the continuation, we get;

$$\begin{aligned}\Big[(s-p_3)\prod_{i=4}^{M+1}(s-p_i).(s-z_1)+\\ (p_3-z_2)\prod_{i=4}^{M+1}(s-p_i).(s-z_1)+\ldots\Big]\end{aligned} \tag{57}$$

This is similar to what we see in Equation 54, and thus we can repeat the steps leading up all the way to the final two;

$$\Big[(s-p_{M+1})\prod_{i=1}^{M-1}(s-z_i)+(p_{M+1}-z_M)\prod_{i=1}^{M-1}(s-z_i)\Big] \tag{58}$$

Which expands and reduces to

$$\prod_{i=1}^{M}(s-z_i) \tag{59}$$

Finally, consider the full equation for X_{M+2};

$$X_{M+2}=\frac{\prod_{i=1}^{M}(s-z_i)}{\prod_{j=1}^{M+2}(s-p_j)}kU \tag{60}$$

Further terms, such that;

$$\begin{aligned}X_{M+2}&=\frac{\prod_{i=1}^{M}(s-z_i)}{\prod_{j=1}^{M+2}(s-p_j)}kU\\ X_{M+3}&=\frac{X_{M+2}}{(s-p_{M+3})}\\ &\vdots\\ X_N&=\frac{X_{N-1}}{(s-p_N)}\\ Y&=X_N\end{aligned} \tag{61}$$

or, in the case that $M=N-2$, then;

$$Y=X_{M+2}=X_N \tag{62}$$

Either way, the solution is trivially;

$$Y=\frac{\prod_{i=1}^{M}(s-z_i)}{\prod_{j=1}^{N}(s-p_i)}kU(s) \tag{63}$$

Case 5, N poles, M zeroes, where $M=N-1$

For category 3 functions, which is slightly different from the above but follows much the same logic. Consider that we solve all but the final 2 equations in the Laplace system of equations;

$$\begin{aligned}X_N&=\frac{(p_N-z_{N-1})}{(s-p_N)}[X_1+X_2+\ldots+X_{N-1}]\\ Y&=[X_1+X_2+\ldots+X_N]\end{aligned} \tag{64}$$

Following the patterns we explored previously; the solution to this set of equations is;

$$\begin{aligned}Y=&\frac{kU}{\prod_{j=1}^{N}(s-p_j)}\times\Big[\prod_{i=2}^{N}(s-p_i)+\\ &(p_2-z_1)\prod_{i=3}^{N}(s-p_i)+\\ &(p_3-z_2)\prod_{i=4}^{N}(s-p_i).(s-z_1)+\\ &(p_4-z_3)\prod_{i=5}^{N}(s-p_i).\prod_{i=1}^{2}(s-z_i)+\ldots+\\ &(p_{N-1}-z_{N-2})(s-p_N).\prod_{i=1}^{N-3}(s-z_i)+\\ &(p_N-z_{N-1})\prod_{i=1}^{N-2}(s-z_i)\Big]\end{aligned} \tag{65}$$

There is one less term in the denominator then previously expected, because there is no denominator for Y. The square brackets simply reduce down to;

$$Y=\frac{kU}{\prod_{j=1}^{N}(s-p_j)}\times\Big[\prod_{i=1}^{N-1}(s-z_i)\Big] \tag{66}$$

Which is equivalent to

$$Y=\frac{\prod_{i=1}^{M}(s-z_i)}{\prod_{j=1}^{N}(s-p_i)}kU(s) \tag{67}$$

Case 6, N poles, M zeroes, where $M=N$

The final case, adds a level of complexity to the algebraic operations, but the result is very similar. Again, consider all but the final 2 equations in the Laplace system of equations as above, but before we have inserted the correct formulation of $\mathbf{C}$ and $\mathbf{D}$;

$$\begin{aligned}X_N&=\frac{(p_N-z_{N-1})}{(s-p_N)}[X_1+X_2+\ldots+X_{N-1}]\\ Y&=[\mathbf{C}_1X_1+\mathbf{C}_2X_2+\ldots+\mathbf{C}_NX_N]+\mathbf{D}U\end{aligned} \tag{68}$$

The formula for Y is similar to Case 5, but with a single additional term $\mathbf{D}=k$. Start by rearranging the $\mathbf{D}$ to the front, and substituting in all the components of $\mathbf{C}$ (also note, for clarity, more elements are shown and the $\mathbf{C}$ terms are in angle brackets "$\langle\rangle$");

$$\begin{aligned}Y=kU+X_1\Big\langle\sum_{i=1}^{N}(p_i-z_i)\Big\rangle+X_2\Big\langle\sum_{i=2}^{N}(p_i-z_i)\Big\rangle+\\ X_3\Big\langle\sum_{i=3}^{N}(p_i-z_i)\Big\rangle+X_4\Big\langle\sum_{i=4}^{N}(p_i-z_i)\Big\rangle+\ldots+\\ X_{N-1}\Big\langle\sum_{i=N-1}^{N}(p_i-z_i)\Big\rangle+X_N\Big\langle p_N-z_N)\Big\rangle\end{aligned} \tag{69}$$

Now substitute in the solutions for all X_i, recognise there is a common factor kU, and multiply through to create the common denominator; $Y =$

$$\frac{kU}{\prod_{j=1}^{N}(s-p_j)} \times \Big[\prod_{i=1}^{N}(s-p_i)+$$
$$\prod_{i=2}^{N}(s-p_i)\Big\langle\sum_{i=1}^{N}(p_i-z_i)\Big\rangle+$$
$$(p_2-z_1)\prod_{i=3}^{N}(s-p_i)\Big\langle\sum_{i=2}^{N}(p_i-z_i)\Big\rangle+$$
$$(p_3-z_2)\prod_{i=4}^{N}(s-p_i).(s-z_1)\Big\langle\sum_{i=3}^{N}(p_i-z_i)\Big\rangle+$$
$$(p_4-z_3)\prod_{i=5}^{N}(s-p_i).\prod_{i=1}^{2}(s-z_i)\Big\langle\sum_{i=4}^{N}(p_i-z_i)\Big\rangle+\ldots+$$
$$(p_{N-1}-z_{N-2})(s-p_N).\prod_{i=1}^{N-3}(s-z_i)\Big\langle\sum_{i=N-1}^{N}(p_i-z_i)\Big\rangle+$$
$$(p_N-z_{N-1})\prod_{i=1}^{N-2}(s-z_i)\Big\langle p_N-z_N\Big\rangle\Big] \tag{70}$$

The order of algebraic operations to eliminate all the p terms inside the square brackets is similar to above, with one additional step after each operation. From the first two terms we can expand all the p_1 to;

$$\begin{aligned}&\Big[(s)\prod_{i=2}^{N}(s-p_i)-(p_1)\prod_{i=2}^{N}(s-p_i)+\\&(p_1)\prod_{i=2}^{N}(s-p_i)-(z_1)\prod_{i=2}^{N}(s-p_i)+\\&\prod_{i=2}^{N}(s-p_i)\Big\langle\sum_{i=2}^{N}(p_i-z_i)\Big\rangle\ldots\Big]\end{aligned} \tag{71}$$

This reduces to

$$\begin{aligned}&\Big[(s-z_1)\prod_{i=2}^{N}(s-p_i)+\\&\prod_{i=2}^{N}(s-p_i)\Big\langle\sum_{i=2}^{N}(p_i-z_i)\Big\rangle\ldots\Big]\end{aligned} \tag{72}$$

This represents the new first and second terms of the square brackets. Eliminating p_2 is done in two steps, as does the rest of the p terms. Consider the second and third term of the square brackets, with $(s-p_2)$ brought to the front ;

$$\begin{aligned}&\Big[\ldots(s-p_2)\prod_{i=3}^{N}(s-p_i)\Big\langle\sum_{i=2}^{N}(p_i-z_i)\Big\rangle+\\&(p_2-z_1)\prod_{i=3}^{N}(s-p_i)\Big\langle\sum_{i=2}^{N}(p_i-z_i)\Big\rangle\ldots\Big]\end{aligned} \tag{73}$$

We can expand the foremost brackets for p_2, which then cancel, and these terms simplify to;

$$\Big[\ldots(s-z_1)\prod_{i=3}^{N}(s-p_i)\Big\langle\sum_{i=2}^{N}(p_i-z_i)\Big\rangle\ldots\Big] \tag{74}$$

And now we can compare this with the first term

$$\begin{aligned}&\Big[(s-z_1)\prod_{i=2}^{N}(s-p_i)+\\&(s-z_1)\prod_{i=3}^{N}(s-p_i)\Big\langle\sum_{i=2}^{N}(p_i-z_i)\Big\rangle\ldots\Big]\end{aligned} \tag{75}$$

Expand for the remaining p_2 terms, and z_2 terms in the angle brackets;

$$\begin{aligned}&\Big[(s)(s-z_1)\prod_{i=3}^{N}(s-p_i)-(p_2)(s-z_1)\prod_{i=3}^{N}(s-p_i)+\\&(p_2)(s-z_1)\prod_{i=3}^{N}(s-p_i)-(z_2)(s-z_1)\prod_{i=3}^{N}(s-p_i)+\\&(s-z_1)\prod_{i=3}^{N}(s-p_i)\Big\langle\sum_{i=3}^{N}(p_i-z_i)\Big\rangle\ldots\Big]\end{aligned} \tag{76}$$

All the p_2 can now be eliminated;

$$\begin{aligned}&\Big[\prod_{i=1}^{2}(s-z_i)\prod_{i=3}^{N}(s-p_i)+\\&(s-z_1)\prod_{i=3}^{N}(s-p_i)\Big\langle\sum_{i=3}^{N}(p_i-z_i)\Big\rangle\ldots\Big]\end{aligned} \tag{77}$$

This two step pattern of eliminating p terms can be repeated until all of them are eliminated, resulting in the square brackets equal to;

$$\Big[\prod_{i=1}^{N}(s-z_i)\Big] \tag{78}$$

And thus it is finally found that

$$Y = \frac{\prod_{i=1}^{N}(s-z_i)}{\prod_{j=1}^{N}(s-p_j)}kU(s) \tag{79}$$

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APPENDIX

A. Simple Reference Examples

Here the reader can find examples of the conversion for some simple cases, as a quick reference tool.

1 Pole, 0 Zeros

$$\frac{Y(s)}{U(s)} = \frac{1}{(s-p_1)}k$$

$$\begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{bmatrix} = \begin{bmatrix} p_1 & k \\ 1 & 0 \end{bmatrix}$$

1 Pole, 1 Zero

$$\frac{Y(s)}{U(s)} = \frac{(s-z_1)}{(s-p_1)}k$$

$$\begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{bmatrix} = \begin{bmatrix} p_1 & k \\ p_1 - z_1 & k \end{bmatrix}$$

2 Poles, 0 Zeros

$$\frac{Y(s)}{U(s)} = \frac{1}{(s-p_1)(s-p_2)}k$$

$$\begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{bmatrix} = \begin{bmatrix} p_1 & 0 & k \\ 1 & p_2 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

2 Poles, 1 Zero

$$\frac{Y(s)}{U(s)} = \frac{(s-z_1)}{(s-p_1)(s-p_2)}k$$

$$\begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{bmatrix} = \begin{bmatrix} p_1 & 0 & k \\ p_2 - z_1 & p_2 & 0 \\ 1 & 1 & 0 \end{bmatrix}$$

2 Poles, 2 Zeros

$$\frac{Y(s)}{U(s)} = \frac{(s-z_1)(s-z_2)}{(s-p_1)(s-p_2)}k$$

$$\begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{bmatrix} = \begin{bmatrix} p_1 & 0 & k \\ p_2 - z_1 & p_2 & 0 \\ \sum_{i=1}^{2}(p_i - z_i) & p_2 - z_2 & k \end{bmatrix}$$

3 Poles, 0 Zeros

$$\frac{Y(s)}{U(s)} = \frac{1}{(s-p_1)(s-p_2)(s-p_3)}k$$

$$\begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{bmatrix} = \begin{bmatrix} p_1 & 0 & 0 & k \\ 1 & p_2 & 0 & 0 \\ 0 & 1 & p_3 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

3 Poles, 1 Zeros

$$\frac{Y(s)}{U(s)} = \frac{(s-z_1)}{(s-p_1)(s-p_2)(s-p_3)}k$$

$$\begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{bmatrix} = \begin{bmatrix} p_1 & 0 & 0 & k \\ p_2 - z_1 & p_2 & 0 & 0 \\ 1 & 1 & p_3 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

3 Poles, 2 Zeros

$$\frac{Y(s)}{U(s)} = \frac{(s-z_1)(s-z_2)}{(s-p_1)(s-p_2)(s-p_3)}k$$

$$\begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{bmatrix} = \begin{bmatrix} p_1 & 0 & 0 & k \\ p_2 - z_1 & p_2 & 0 & 0 \\ p_3 - z_2 & p_3 - z_2 & p_3 & 0 \\ 1 & 1 & 1 & 0 \end{bmatrix}$$

3 Poles, 3 Zeros

$$\frac{Y(s)}{U(s)} = \frac{(s-z_1)(s-z_2)(s-z_3)}{(s-p_1)(s-p_2)(s-p_3)}k$$

$$\begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{bmatrix} = \begin{bmatrix} p_1 & 0 & 0 & k \\ p_2 - z_1 & p_2 & 0 & 0 \\ p_3 - z_2 & p_3 - z_2 & p_3 & 0 \\ \sum_{i=1}^{3}(p_i - z_i) & \sum_{i=2}^{3}(p_i - z_i) & p_3 - z_3 & k \end{bmatrix}$$

B. SISO Transfer Functions with n poles and $n-1$ zeros

$$\frac{Y(s)}{U(s)} = \frac{b_1 s^{n-1} + \ldots + b_{n-1}s + b_n}{s^n + a_1 s^{n-1} + \ldots + a_{n-1}s + a_n}$$

Controllable Canonical Form:

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & & 0 & 0 \\ \vdots & & \ddots & & \vdots \\ 0 & 0 & & 1 & 0 \\ 0 & 0 & \ldots & 0 & 1 \\ -a_n & -a_{n-1} & \ldots & -a_2 & -a_1 \end{bmatrix} \qquad \mathbf{B} = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

$$\mathbf{C} = \begin{bmatrix} b_n & b_{n-1} & \ldots & b_2 & b_1 \end{bmatrix} \qquad \mathbf{D} = \begin{bmatrix} 0 \end{bmatrix}$$

Observable Canonical Form:

$$\mathbf{A} = \begin{bmatrix} 0 & \ldots & 0 & 0 & -a_n \\ 1 & & 0 & 0 & -a_{n-1} \\ & \ddots & & \vdots & \vdots \\ 0 & & 1 & 0 & -a_2 \\ 0 & \ldots & 0 & 1 & -a_1 \end{bmatrix} \qquad \mathbf{B} = \begin{bmatrix} b_n \\ b_{n-1} \\ \vdots \\ b_2 \\ b_1 \end{bmatrix}$$

$$\mathbf{C} = \begin{bmatrix} 0 & \ldots & 0 & 0 & 1 \end{bmatrix} \qquad \mathbf{D} = \begin{bmatrix} 0 \end{bmatrix}$$

Diagonal Canonical Form: Factorize denominator, compute partial fractions;

$$\begin{aligned} \frac{Y(s)}{U(s)} &= \frac{b_1 s^{n-1} + \ldots + b_{n-1}s + b_n}{(s-p_1)(s-p_2)...(s-p_n)} \\ &= \frac{r_1}{(s-p_1)} + \frac{r_2}{(s-p_2)} + ... + \frac{r_n}{(s-p_n)} \end{aligned}$$

$$\mathbf{A} = \begin{bmatrix} p_1 & 0 & \ldots & 0 & 0 \\ 0 & p_2 & & 0 & 0 \\ \vdots & & \ddots & & \vdots \\ 0 & 0 & & p_{n-1} & 0 \\ 0 & 0 & \ldots & 0 & p_n \end{bmatrix} \qquad \mathbf{B} = \begin{bmatrix} r_1 \\ r_2 \\ \vdots \\ r_{n-1} \\ r_n \end{bmatrix}$$

$$\mathbf{C} = \begin{bmatrix} 1 & 1 & \ldots & 1 & 1 \end{bmatrix} \qquad \mathbf{D} = \begin{bmatrix} 0 \end{bmatrix}$$

Pole-Zero Difference Form: Factorize the numerator and denominator;

$$\frac{Y(s)}{U(s)} = \frac{(s-z_1)(s-z_2)...(s-z_{n-1})}{(s-p_1)(s-p_2)...(s-p_n)}k$$

$$\mathbf{A} = \begin{bmatrix} p_1 & 0 & 0 & & 0 \\ \Delta_{21} & p_2 & 0 & \ldots & 0 \\ \Delta_{32} & \Delta_{32} & p_3 & & 0 \\ \vdots & & & \ddots & \\ \Delta_{n,n-1} & \Delta_{n,n-1} & \Delta_{n,n-1} & \ldots & p_n \end{bmatrix} \qquad \mathbf{B} = \begin{bmatrix} k \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix}$$

$$\mathbf{C} = \begin{bmatrix} 1 & 1 & 1 & \ldots & 1 \end{bmatrix} \qquad \mathbf{D} = \begin{bmatrix} 0 \end{bmatrix}$$

where $\Delta_{ji} = p_j - z_i$

C. SISO Transfer Functions with n poles and n zeros

$$\frac{Y(s)}{U(s)} = \frac{b_0 s^n + b_1 s^{n-1} + \ldots + b_{n-1} s + b_n}{s^n + a_1 s^{n-1} + \ldots + a_{n-1} s + a_n}$$

Controllable Canonical Form:

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & & 0 & 0 \\ \vdots & & \ddots & & \vdots \\ 0 & 0 & & 1 & 0 \\ 0 & 0 & \ldots & 0 & 1 \\ -a_n & -a_{n-1} & \ldots & -a_2 & -a_1 \end{bmatrix} \qquad \mathbf{B} = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

$$\mathbf{C} = \begin{bmatrix} (b_n - a_n b_0) & (b_{n-1} - a_{n-1} b_0) & \ldots & (b_2 - a_2 b_0) & (b_1 - a_1 b_0) \end{bmatrix} \qquad \mathbf{D} = \begin{bmatrix} b_0 \end{bmatrix}$$

Observable Canonical Form:

$$\mathbf{A} = \begin{bmatrix} 0 & \ldots & 0 & 0 & -a_n \\ 1 & & 0 & 0 & -a_{n-1} \\ & \ddots & & \vdots & \vdots \\ 0 & & 1 & 0 & -a_2 \\ 0 & \ldots & 0 & 1 & -a_1 \end{bmatrix} \qquad \mathbf{B} = \begin{bmatrix} b_n - a_n b_0 \\ b_{n-1} - a_{n-1} b_0 \\ \vdots \\ b_2 - a_2 b_0 \\ b_1 - a_1 b_0 \end{bmatrix}$$

$$\mathbf{C} = \begin{bmatrix} 0 & \ldots & 0 & 0 & 1 \end{bmatrix} \qquad \mathbf{D} = \begin{bmatrix} b_0 \end{bmatrix}$$

Diagonal Canonical Form: Factorize denominator, compute partial fractions;

$$\frac{Y(s)}{U(s)} = \frac{b_0 s^n + b_1 s^{n-1} + \ldots + b_{n-1} s + b_n}{(s-p_1)(s-p_2)\ldots(s-p_n)}$$
$$= b_0 + \frac{r_1}{(s-p_1)} + \frac{r_2}{(s-p_2)} + \ldots + \frac{r_n}{(s-p_n)}$$

$$\mathbf{A} = \begin{bmatrix} p_1 & 0 & \ldots & 0 & 0 \\ 0 & p_2 & & 0 & 0 \\ \vdots & & \ddots & & \vdots \\ 0 & 0 & & p_{n-1} & 0 \\ 0 & 0 & \ldots & 0 & p_n \end{bmatrix} \qquad \mathbf{B} = \begin{bmatrix} r_1 \\ r_2 \\ \vdots \\ r_{n-1} \\ r_n \end{bmatrix}$$

$$\mathbf{C} = \begin{bmatrix} 1 & 1 & \ldots & 1 & 1 \end{bmatrix} \qquad \mathbf{D} = \begin{bmatrix} b_0 \end{bmatrix}$$

Pole-Zero Difference Form: Factorize the numerator and denominator;

$$\frac{Y(s)}{U(s)} = \frac{(s-z_1)(s-z_2)\ldots(s-z_n)}{(s-p_1)(s-p_2)\ldots(s-p_n)} k$$

$$\mathbf{A} = \begin{bmatrix} p_1 & 0 & 0 & & 0 \\ \Delta_{21} & p_2 & 0 & \ldots & 0 \\ \Delta_{32} & \Delta_{32} & p_3 & & 0 \\ \vdots & & & \ddots & \\ \Delta_{n,n-1} & \Delta_{n,n-1} & \Delta_{n,n-1} & \ldots & p_n \end{bmatrix} \qquad \mathbf{B} = \begin{bmatrix} k \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix}$$

$$\mathbf{C} = \begin{bmatrix} (\sum_{i=1}^{n} \Delta_{ii}) & (\sum_{i=2}^{n} \Delta_{ii}) & \ldots & (\Delta_{n-1,n-1} + \Delta_{nn}) & (\Delta_{nn}) \end{bmatrix} \qquad \mathbf{D} = \begin{bmatrix} k \end{bmatrix}$$

where $\Delta_{ji} = p_j - z_i$

Note: An alternative description of the elements of $\mathbf{C}$, is that they are the sum of the elements in $\mathbf{A}$ from the same column minus the final zero z_n.

D. Matlab Script for Proof by Example

The following script constructs the matrices $\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D}$ per the Pole-Zero-Difference Form using randomly assigned values of p, z, k. It then compares the results for the transfer-function and state-space using Matlab's Control System Toolbox.

```
N = 10; M = 10;  k = rand(1);
p = -sort(rand(1,N),'descend'); z = sort(rand(1,M)-0.5,'descend');

B = [k;zeros(N-1,1)];  C = ones(1,N); D = 0;

Atril = diag(ones(N-1,1),-1);
for i = 1:min(M,N-1)
    Atril(i+1,:) = p(i+1)-z(i);
end
if M<N-1
    Atril(M+2,:)=1;    C = [zeros(1,N-1),1];
end
A = diag(p) + tril(Atril,-1);
if M==N
    C = sum(A)-z(end);  D = k;
end

func1 = zpk(z,p,k);  step(func1,'b');   hold on
func2 = ss(A,B,C,D); step(func2,'r--');

zcalc = eig([A B;C D],diag([ones(1,length(p)) 0]));
zind = find(isinf(zcalc)); zcalc(zind)=[];
disp('Zeros input vs calculated;')
disp([z' sort(zcalc,'descend')])
```